Linear Algebra II

Mid-term Examination

Instructions: All questions carry ten marks. Vector spaces are assumed to be finite dimensional.

1. Let $T : V \to V$ be a linear operator on vector space of dimension n over a field F. Assume that T has n *distinct* eigen values in F. Prove that there exists a basis $\mathcal{B}$ of V with respect to which the matrix of T is a diagonal matrix.

2. Let T_1, T_2 be linear operators on a vector space V over a field F such that $T_1 \circ T_2 = T_2 \circ T_1$. Let V_λ denote the subspace of V consisting of vectors v such that $T(v) = \lambda v$. Prove that V_λ is T_2-invariant. If $V_\lambda \neq \{0\}$, then will it have to contain eigen vectors for T_2? Justify your answer.

3. Let V be a vector space over $\mathbb{C}$. Let $<,>$ be a Hermitian form on V. Prove that it is positive definite if and only if there exists an orthonormal basis of V with respect to $<,>$.

4. Define Hermitian matrix. Prove that a complex matrix of order n is Hermitian if and only if X^*AX is a real number for every column vector $X \in \mathbb{C}^n$.

5. Let
$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$
Find an orthonormal basis of $\mathbb{R}^2$ with repect to the bilinear form given by A.